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A MULTIVARIATE APPROACH TO TESTS OF HYPOTHESES ASSOCIATED WITH THE ELECTROMAZE

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Project NR153-148: Report No. 4- - Errors

Page 6 Equation (7) should read:

$$\begin{Bmatrix} y^i_{11} & \cdot & \cdot & \cdot & y^i_{12} \\ \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & & \cdot \\ y^i_{r_1 1} & \cdot & \cdot & \cdot & y^i_{r_2 2} \end{Bmatrix}$$

Page 6 Equation (9) should read: $\left\{ y^i_{a_2 1} \right\}$ are MNID (μ^i_2, σ_{ij})

Page 7 Equation (13), lines 5 and 9 should read:

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

Page 8 Equation (20), line 5 should read:

$$= r_1 \left\{ y^i_{.1} \right\} \left\{ y^i_{.1} \right\} + r_2 \left\{ y^i_{.2} \right\} \left\{ y^i_{.2} \right\}$$

Page 12 Table 3, SS and XP should read:

$$\begin{Bmatrix} 17265.96 & 5479.38 \\ 5479.38 & 1738.89 \end{Bmatrix}$$

$$\begin{Bmatrix} 111411.24 & 198477.42 \\ 198477.42 & 1328007.69 \end{Bmatrix}$$

Page 14 Table 4 should read: Statistic / Problem

Page 17 line 14: Correct the word interpretation.

Page 17 Reference 3 should read: "The Statistical Interpretation of Degrees of Freedom"

A Multivariate Approach to Tests of Hypotheses Associated with the Electromaze

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Abstract

An application of multivariate analysis of variance is made to seven electromaze problems which were given to physics and American Studies majors at the University of Minnesota. Reasons and advantages for considering a multivariate hypothesis are given and a discussion is included which contrasts matrix variates and linear variates. The actual test used is derived from elementary principles and is known more commonly as Hotelling's T^2. Seven multivariate analyses on two variates —trials and time— were made and the results summarized in a convenient table together with the seven separate analyses for both variates considered independently.

A Multivariate Approach to Tests of Hypotheses Associated with the Electromaze*

Almost from the beginning of experience with the electromaze, the choice of the proper criterion by which performance should be judged needed careful consideration. Kruglak (1) was the first to recognize the problem and in his exploratory studies decided to consider two criteria. The data were accordingly analyzed, using the two criteria, but they were treated as independent variates. This supposition was shown untenable and Kruglak suggested an investigation that would show how the criteria could be considered jointly and what the results of such analyses would be.

Before getting into the details of the analysis a brief discussion will be given which tells why the usual univariate analysis of variance model is not appropriate for the electromaze data.

Needs for an Extension of the Customary Models

Although the procedures of ordinary analysis of variance and covariance are extremely useful in research, they are not sufficient for the analysis of many experiments. Even when the principles of analysis of variance (anova) are applicable to data, this insufficiency may arise from one of two circumstances:

1. The incapability of one dependent variate to describe satisfactorily, or to define, the concept which the researcher wishes to consider in an experiment.
2. The incapability of independent tests of significance to assign correctly exact probabilities of rejection or acceptance of hypotheses when these tests are applied successively to variates measured on the same subjects or objects.

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These two criticisms may be illustrated by considering some hypotheses associated with the electromaze. It was the original intention of Kruglak (1) to contrast the performance of two groups of students, designated as "Physics Majors" and "American Studies Majors" on each of seven problems as presented by the electromaze. The characteristics of the groups are not of immediate concern to us, but the variate to be analyzed is. The researcher was interested in testing a null hypothesis by means of a variate which may be termed "Ability to solve problems on the electromaze." This variate has been considered by the participants and capable, critical judges to be related to"logical procedure" and/or "scientific method."

The subjects "ability to solve problems on the electromaze" must be measured by some objective evidence. The two measures, suggested by Kruglak, are the <u>number of trials</u> and the <u>time in seconds</u> which were needed by the subjects to solve the problems. When considered separately or independently, neither of these measures appears to be satisfactory for measuring the variate under study. This can be justified by considering two extreme situations. Suppose a student does not possess knowledge of "logical procedures" or the "scientific method." This student when asked to solve a problem presented by the electromaze would attack the problem somewhat haphazardly and would punch the keys somewhat indiscriminately. Such punching could solve the problem in a relatively short time if the problem were simple enough, but it is expected that, in general, it would take a considerable number of punches to do so. Alternatively, consider a second type of student who possesses knowledge of certain logical prodecures and/or the "scientific method." It will take some time for this student to formulate his attack on the problem, without having made a move. However, it is probable that fewer trials per unit time would be made in this case than by the haphazard approach. If several students of the "haphazard" type were to solve a problem on the electromaze, their total number of trials would be larger, but their time of solution would be less, than for the same number of students of the second type who had solved the same problem (or perhaps about equal to the first type if the problem was very simple). Observations of students actually performing on the electromaze point to the fact that they use a mixture of the rational and random approach to a varying degree. If the above argument is permitted it would seem that in general the variates, number of trials and time in seconds, need to be considered simultaneously in the analysis which purports to test hypotheses regarding the variate "ability to solve problems on the electromaze." The question now naturally arises on how to analyze these variates together.

Linear Variates and Matrix Variates

A common method of handling this problem is to construct what may be termed a linear variate or linear definition of ability to solve problems on the electromaze. That is, the two variates are compounded by a linear form. The coefficients of such forms are variously determined by guesswork, shrewd judgment, or by some mathematical criterion. Anyone of these procedures yields a new compound variate for each subject of the experiment. These new indices are the variables which are analyzed. Another common method of analysis for situations of this type would be to make two separate analyses, one for each variate. This procedure, besides involving interpretational problems, in complicated by the second objection listed on page 1 and will be discussed later.

Linear variates, although commonly used, have two serious objections associated with them. If the variates are simply added together with the weight for each variate equal to unity, i.e., unweighted, the composite score may not be representative of the set if there is more than one factor involved, which is usually the case. Such a combination is often uncritically done by researchers in psychology and education, particularly if the variates involved have the same dimensions. As the problem here concerns variates with dimensions of trials and seconds, we should be skeptical of such practices. Moreover, if such additions were effected, what units would be assigned to the new variate?

An alternative to the linear variate approach and the one used in this paper would be to deal with the variates as aggregates which are not synthesized in any arbitrary or regulatory manner. To do this let the variates be elements of a matrix wherein their natural properties assume their rightful values. We shall speak of the aggregate we are discussing, i.e., ability to solve problems on the electromaze, as a matrix variate. For some problems it may be decreed that the elements specify a concept so adequately that the matrix variate is more properly called a matrix definition. The logical scientific constructs of education and psychology may be considered to be matrix definitions, and hypotheses regarding them may be treated by the method —or its generalizations— shown in this paper.

Multivariate Analysis of Variance and Probability Considerations

As stated before, it is the purpose of this paper to describe the theoretical and analytical aspects which underlie tests of hypotheses that involve the second order matrix variate, "ability to solve problems on the electromaze." The methodology to be presented constitutes a very small part of what is currently known as multivariate analysis in statistics. The particular methodology presented here is a direct multivariate generalization of the univariate analysis of variance technique commonly called Fisher's t-test. First principles, review of literature, and examples of this topic and many others which collectively are called multivariate analysis of variance or multanova are given in great detail elsewhere (2).

The other criticism on page 1 of the use made of univariate anova was concerned with tests of significance. Suppose we are analyzing the data of an experiment which has a matrix variate of order two. We are interested in testing some hypothesis or hypotheses. Until we know how to handle this problem, we will have to be content to analyze the data astwo univariate experiments. The hypothesis we want to test is, say,

(1) $$\mu_A^i = \mu_B^i = \mu_C^i \qquad i = 1, 2$$

Thus we have three treatmants, A, B, and C, and we would like to know if the population means are equal for variates 1 and 2. For lack of a more optimum method, we might proceed with one or the other of the following alternatives:

1. We could reject an hypothesis which is made on the matrix variate if we reject it for either variate, 1 or 2.
2. We could reject an hypothesis which is made on the matrix variate if we reject it for both variates 1 and 2.

If R_H is the probability of rejecting the hypothesis on the matrix variate then our first alternative involves the probability formula:

(2) $$P(R_H) = P(R_1) + P(R_2) - P(R_1R_2)$$

where $P(R_1)$ and $P(R_2)$ are the probabilities of the hypothesis for variate 1 alone and 2 alone, respectively. If variates 1 and 2 are independently distributed then

(3) $$P(R_1R_2) = P(R_1)P(R_2)$$

Assume $P(R_1) = P(R_2) = .05$ and 1 and 2 are independent, then

(4) $$P(R_H) = .05 + .05 - .0025 = .0975$$

Thus if the above assumptions hold, the Type I error involved in testing the matrix variate hypothesis is slightly less than .10.

The second alternative involves the formula:

(5) $$P(R_H) = P(R_1)\ P(R_2)$$

if we assume variates 1 and 2 as independent. Further if $P(R_1) = P(R_2) = .05$, then

(6) $$P(R_H) = (.05)(.05) = .0025$$

This value is the Type I error for the second alternative and is quite discrepant to the value .0975 obtained by the first alternative method. If we desire to make $P(R_H) = .05$ for either alternative we can do so by changing

$P(R_1)$ or $P(R_2)$ both. This is a simple matter when there exists independence between variates 1 and 2; otherwise the situation can be quite complicated. When the variates are not independent, their multivariate distribution must be considered. Even an assumption of a multivariate normal distribution would not be of value when testing the hypothesis by either or both of the suggested alternatives unless rho is known exactly.

What is needed then is an optimum method by which we can test hypotheses using matrix variates, and which has significant tests that may be interpreted exactly in terms of probability without recourse to nuisance parameters. Such a methodology is <u>multanova</u>. A special case of this methodology will now be presented and illustrated by means of the electromaze data.

The Two-Sample Problem or the Multivariate Generalization of Fisher's t

Let

(7)
$$\left\{\begin{matrix} y^i_{11} & \cdot & \cdot & \cdot & y^i_{12} \\ \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & & \cdot \\ y_{r_1 1} & \cdot & \cdot & \cdot & y_{r_2 2} \end{matrix}\right\} \qquad i = 1,\ldots,P$$

be $r_1 + r_2$ random observations on $\underline{p}$ variates. Note that the superscript $\underline{i}$ is not a power, but merely a designate of the variate to which the variable belongs. Assume further that

(8) $\left\{y^i_{a_1 1}\right\}$ are MNID (μ^i_1, σ_{ij}) $a_1 = 1, \ldots, r_1$

(MNID = multivariately normally and independently distributed)

(9) $\left\{y_{a_2 1}\right\}$ are MNID (μ^i_2, σ_{ij}) $a_2 = 1, \ldots, r_2$
$i, j = 1, \ldots, p$

or that

(10) $\left\{y_{a^i_k k}\right\}$ are MNID (μ^i_2, σ_{ij}) $k = 1, 2;\ i, j = 1, \ldots, p$

In this problem we wish to test the multivariate null hypothesis

(11) $\left\{\mu^i_1 = \mu^i_2\right\}$ $\qquad i = 1, \ldots, p$

The mathematical observation model is the mean of the multivariate normal distribution of the observations. Their expectations may be written as

(12) $$E\{y_{ak}^{i}k\} = \{\mu_k^i\}$$

We may write out the expectation for each matrix observation as:

$$\begin{aligned}
E\{y_{11}^{i}\} &= 1\{\mu_1^i\} + 0\{\mu_2^i\}\\
E\{y_{21}^{i}\} &= 1\{\mu_1^i\} + 0\{\mu_2^i\}\\
&\vdots\\
E\{y_{r_1 1}^{i}\} &= 1\{\mu_1^i\} + 0\{\mu_2^i\}\\
E\{y_{12}^{i}\} &= 0\{\mu_1^i\} + 1\{\mu_2^i\}\\
E\{y_{22}^{i}\} &= 0\{\mu_1^i\} + 1\{\mu_2^i\}\\
&\vdots\\
E\{y_{r_2 2}^{i}\} &= 0\{\mu_1^i\} + 1\{\mu_2^i\}
\end{aligned} \qquad (13)$$

The coefficients of the parameters constitute the estimation space and this space is generated by two vectors:

(14) $$\alpha_1 = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$$
$$\alpha_2 = (0, 0, \ldots, 0, 1, 1, \ldots, 1)$$

The rank of this vector space is two which signifies there are two multivariate estimates for the parameters. The least squares normal equations are:

(15) $$\{\mu_1^i\}(\alpha_1 \cdot \alpha_1) + \{\mu_2^i\}(\alpha_1 \cdot \alpha_2) = \{\alpha_1 \cdot \eta^i\}$$
$$\{\mu_1^i\}(\alpha_2 \cdot \alpha_1) + \{\mu_2^i\}(\alpha_2 \cdot \alpha_2) = \{\alpha_2 \cdot \eta^i\}$$

where

(16) $$\{\eta^i\} = \{y_{11}^i, y_{21}^i, \ldots, y_{r_1 1}^i, y_{12}^i, y_{22}^i, \ldots, y_{r_2 2}^i\}$$

because

(17) $$(\alpha_1 \cdot \alpha_1) = r_1 \text{ and } (\alpha_2 \cdot \alpha_2) = r_2$$
$$(\alpha_1 \cdot \alpha_2) = 0 \text{ and } (\alpha_2 \cdot \alpha_1) = 0$$
$$\{\alpha_1 \cdot \eta^i\} = \Big\{\sum_{a_1=1}^{r_1} y_{a_1 1}^i\Big\} \text{ and } \{\alpha_2 \cdot \eta^i\} = \Big\{\sum_{a_2=1}^{r_2} y_{a_2 2}^i\Big\}$$

We solve the normal equations as

(18) $$\{\hat{\mu}_1^i\} = \{y_{.1}^i\} \text{ and } \{\hat{\mu}_2^i\} = \{y_{.2}^i\}$$

Which are the two multivariate sample means. The total sums of squares (SS) and cross-products (XP) of the observations are

(19) $$SST_{ij} = \sum_{k=1}^{2}\sum_{a_k=1}^{r_k} \left\{ y\ a_k^i k\right\}\left\{ y\ a_k^j k\right\}$$

where the j matrix is the transpose of the i matrix.

The SS and XP due to estimates is

(20) $$SSE_{ij} = \left\{y._1^i\right\}\left\{\alpha_1 \cdot \eta^i\right\} + \left\{y._2^i\right\}\left\{\alpha_2 \cdot \eta^i\right\}$$
$$= r_1\left\{y._1^i\right\}\left\{y._1^i\right\} + r_2\left\{y._2^i\right\}\left\{y._2^j\right\}$$

Since it is known that the SS and XP for error is

(21) $$SSe_{ij} = SST_{ij} - SSE_{ij} = A$$
$$= \sum_{k=1}^{2}\sum_{a_k=1}^{r_k}\left\{ y a_k^i k - y_{\cdot k}^i\right\}\left\{y a_k^j k - y_{\cdot k}^j\right\}$$
$$= \sum_{a_1=1}^{r_1}\left\{ y a_1^i 1 - y_{\cdot 1}^i\right\}\left\{y a_1^j - y_{\cdot 1}^j\right\} + \sum_{a_2=1}^{r_2}\left\{ y a_2^i 2 - y_{\cdot 2}^i\right\}\left\{y a_2^j 2 - y_{\cdot 2}^j\right\}$$

which is called the _generalized_ _variance-covariance_ (_varcovar_) _matrix_. Incidentally, this is a familiar form when $i = j = 1$.

Next we need to find the SS and XP due to our hypothesis so that it may be tested. The hypothesis again is

(22) $$\left\{\mu_1^i\right\} = \left\{\mu_2^i\right\}$$

A linear function of the population means is called an _hypothesis_ _contrast_ if the sum of the coefficients is zero. Thereby

(23) $$c_1\left\{\mu_1^i\right\} + c_2\left\{\mu_2^i\right\}$$

is a contrast if $\sum_{k=1}^{2} c_k = 0$. The hypothesis may be stated in the form of the contrast

(24) $$\left\{\mu_1^i\right\} - \left\{\mu_2^i\right\} = \left\{0^i\right\}$$

The sample estimates of the hypothesis appear to carry k degrees of freedom (see references 2 and 3), but the contrast itself is a linear restriction on the estimates so the degrees of freedom carried by the hypothesis contrast is k-1. For our particular case this number is 1 for k=2. The sample estimate of the hypothesis contrast is given by

(25) $$\left\{y_{\cdot 1}^i\right\} - \left\{y_{\cdot 2}^i\right\} = \left\{0^i\right\}$$

This equation is orthogonal to the general multivariate mean

(26) $$\left\{y_{\cdot\cdot}^i\right\} = \left\{\frac{r_1\ y_{\cdot 1}^i + r_2 y_{\cdot 2}^i}{r_1 + r_2}\right\}$$

and the SS and XP associated with this function is

(27) $$(r_1+r_2)\left\{y_{..}^{i}\right\}\left\{y_{..}^{j}\right\}$$

Consequently the SS and XP of the one degree of freedom carried by the hypothesis contrast is

(28) $$B = (20) - (27)$$
$$= r_1\left\{y_{.1}^{i} - y_{..}^{i}\right\}\left\{y_{.1}^{j} - y_{..}^{j}\right\} + r_2\left\{y_{.2}^{i} - y_{..}^{i}\right\}\left\{y_{.2}^{j} - y_{..}^{j}\right\}$$

This SS and XP is used together with A of the criterion statistic

(29) $$\left| B - \theta\,(A + B)\right| = 0$$

where A and B have independent Wishart distributions $W(a_{ij}, \sigma_{ij}, p, r_1+r_2-2)$ and $W(b_{ij}, \sigma_{ij}, p, 1)$ respectively. The complete multivariate analysis of variance is summarized in Table 1.

Table 1

The Multivariate Analysis of Variance Table for the Generalization of Fisher's t

Source of Variation	D. F.	SS and XP	Criterion
Hypothesis:			
Between samples	1	B = (28).	θ_1
Error:			
Within samples	r_1+r_2-2	A = (21)	

Because the rank of B is unity, there will be only one determinamental root independently of the value of p. The criterion given in (29) is not the only one used for this problem. Hotelling, in giving the original solution for this problem, defined a statistic T^2 for testing hypothesis (24). T^2 was defined as

(30) $$T^2 = \left\{\delta^{i}\right\}\left\{s^{-1}\right\}\left\{\delta^{j}\right\}$$

where

(31) $$\left\{ \delta^{i} \right\} = \sqrt{r_1 + r_2} \left\{ y_{.1}^{i} - y_{.2}^{i} \right\}$$

and $\left\{ s^{-1} \right\}$ is the product of the inverse matrix of A and f, the degrees of freedom of A, i.e., $r_1 + r_2 - 2 = f$. Hotelling found the distribution of T^2 to be an incomplete beta distribution which by the transformation

(32) $$F = \frac{(f-p+1)}{fp} \; T^2$$

becomes the variance ratio distribution of Fisher for $n_1 = p$ and $n_2 = f-p+1$. The relationship between θ and T^2 is given by

(33) $$\theta_1 = \frac{T^2}{f+T^2} \quad \text{and } T^2 = \frac{f\theta_1}{1-\theta_1}$$

Therefore,

(34) $$F(f - {}^{p}_{p} + 1) = \frac{f - p + 1}{p} \quad \frac{\theta_1}{1 - \theta_1}$$

The Application of the Multivariate Generalization of Fisher's t to Electromaze Data

A complete problem will be illustrated for the first problem and the results of the other six problems will be summarized. Let

$y^{1}_{a_1 1}$ be the <u>number of trials</u> made in solving the problem by the a_1th Physics student

$y^{2}_{a_1 1}$ be the <u>time</u> in seconds required to solve the problem by the a_1th Physics student

$y^{1}_{a_2 2}$ be the <u>number of trials</u> made in solving the problem by the a_2th American Studies student

$y^{2}_{a_2 2}$ be the <u>time</u> in seconds required to solve the problem by the a_2th American Studies student

The complete data are given in Table 2.

Table 2

Summary Table for the Generalized Fisher's t Illustration

Major	Variate	Σ	SS and XP 1	SS and XP 2
Physics	1	642	34,978	136,032
(r_1=25)	2	2056		1,299,262
American Studies	1	1302	177,680	238,608
(r_2=20)	2	1895		377,381

It is assumed that

(33) $$\{y_{a_k^i k}\} \text{ are MNID } (\mu_k^i, \sigma_{ij})$$

and we wish to test the hypothesis

(34) $$\begin{Bmatrix} \mu_1^1 \\ \mu_1^2 \end{Bmatrix} = \begin{Bmatrix} \mu_2^1 \\ \mu_2^2 \end{Bmatrix}$$

The equations of expectation are given by (13) where r_1= 25 and r_2=20 and the estimation space is generated by (14). Solving the normal equations (15), we find

(35) $$\begin{Bmatrix} \hat{\mu}_1^1 \\ \hat{\mu}_1^2 \end{Bmatrix} = \begin{Bmatrix} 25.68 \\ 82.24 \end{Bmatrix} \text{ and } \begin{Bmatrix} \hat{\mu}_2^1 \\ \hat{\mu}_2^2 \end{Bmatrix} = \begin{Bmatrix} 65.10 \\ 94.75 \end{Bmatrix}$$

which are the sample means for each variate for each group of subjects. The total SS and XP are found from the entries in Table 2. Corresponding to (20) we find

(36) $$\begin{aligned} SSE_{ij} &= \begin{Bmatrix} 25.68 \\ 82.24 \end{Bmatrix} \{642 \quad 2056\} + \begin{Bmatrix} 65.10 \\ 94.75 \end{Bmatrix} \{1302 \quad 1895\} \\ &= \begin{Bmatrix} 16486.56 & 52799.08 \\ 52798.08 & 169085.44 \end{Bmatrix} + \begin{Bmatrix} 84760.20 & 123364.50 \\ 123364.50 & 179551.25 \end{Bmatrix} \\ &= \begin{Bmatrix} 101246.76 & 176162.58 \\ 176162.58 & 348636.69 \end{Bmatrix} \end{aligned}$$

Using (21) we get

$$SSe_{ij} = \begin{Bmatrix} 212658 & 374640 \\ 374640 & 1676643 \end{Bmatrix} - \underbrace{\begin{Bmatrix} 101246.76 & 176162.58 \\ 176162.58 & 348636.69 \end{Bmatrix}}_{= A} = \begin{Bmatrix} 111411.24 & 198477.42 \\ 198477.42 & 1328007.69 \end{Bmatrix}$$

We yet need to find the SS and XP due to the hypothesis. These values are found by using (28), but we need to calculate (27) which is

$$(38) \qquad 45 \begin{Bmatrix} 43.20 \\ 87.80 \end{Bmatrix} \begin{Bmatrix} 43.20 & 87.80 \end{Bmatrix} = \begin{Bmatrix} 83980.80 & 170683.20 \\ 170683.20 & 346897.80 \end{Bmatrix}$$

Then,

$$(39) \qquad B = (20) - (27) = \begin{Bmatrix} 17265.96 & 5479.38 \\ 5479.38 & 1738.89 \end{Bmatrix}$$

The criterion, (29), becomes

$$(40) \qquad \begin{vmatrix} 17265.96 - \theta\,(128677.20) & 5479.38 - \theta\,(203956.80) \\ 5479.38 - \theta\,(203956.80) & 1738.89 - \theta\,(1329746.58) \end{vmatrix} = 0$$

Solving for θ, we get

$$(41) \quad 129509690357.7360\ \theta^2 - 209479931.1568\ \theta = 0$$

$$\theta = .1617,\ 0$$

We now test the significance of the non-zero root, θ, with equation (34).

$$(42) \qquad F_{\binom{2}{42}} = \frac{43 - 2 + 1}{2} \; \frac{(.1617)}{(.8383)} = 4.05$$

which is significant at the .05 level. If we evaluate T^2 we find it to be 8.29. The analysis may be summarized as shown in Table 3.

Table 3

The Multivariate Analysis of Variance Table for the Illustration of the Generalization of Fisher's t

Source of Variation	D. F.	SS and XP		Criterion
Hypothesis:				
Between Samples	1	17265.96 5479.38	5479.38 1738.89	$\theta_1 = .1617$*
Error:				
Within Samples	43	111411.24 198477.42	198477.42 1328007.69	

*means significant at .05

Had we chosen either one of the criteria variates —**trials** or time— as the variate upon which the analysis was to be made, different conclusions would have been reached. From the diagonal elements of the matrices of Table 3 we can construct the analysis of variance which would be made on each variate separately by using Fisher's t. Thus,

$$F_1 \left(\tfrac{1}{43}\right) = t_1^2 = \frac{(43)(17265.96)}{111411.24} = 6.66^* \tag{43}$$

$$F_2 \left(\tfrac{1}{43}\right) = t_2^2 = \frac{(43)(1738.89)}{1328007.69} = .06 \tag{44}$$

the results show that the groups are not homogeneous with respect to the mean number of trials, but they are homogeneous with respect to the time involved in the solution. It is not too surprising that the greater mean values belong to the American Studies group. Here greater mean values indicate less proficiency at the task.

The summary analysis of Electromaze problem 1 and the other six problems is given in Table 4#.

Table 4

The Results of Multivariate and Univariate Tests of Significance Associated with Seven Problems on the Electromaze

Statistic / Problem	θ_1	T^2	$F(^2_{42})$	$F_1(^1_{43})$	$F_2(^1_{43})$
1	.1617	8.29	4.05*	6.66*	.06
2	.1652	8.51	4.16*	8.08**	.35
3	.0013	.06	.03	.05	.05
4	.0091	.39	.19	.03	.01
5	.0487	2.20	1.08	.00	1.29
6	.0985	4.70	2.06	.46	.53
7	.0250	1.10	.54	.00	.70

*significant at .05 level　　Variate 1 = Trials

**significant at .01 level　　Variate 2 = Time

The author would like to thank Mrs. Leah Horwitz who carried out these calculations with her usual complete accuracy and unusual speed.

We note from Table 4 that there exist significant differences on the first and second problems on the multivariate test and for the trials variate on the univariate tests. Such results are to be expected since the problems were very simple and could be solved in a short time whether or not the "scientific method" or "logical procedures" were used to effect the solution. For more difficult problems, especially 5, 6, and 7, the time factor became increasingly important but not significantly so. If we are to believe that the electromaze measures the application of "scientific method" and/or "logical procedure," then a possible interpretation of the fact that the variate "number of trials" discriminates for the very simple problems and not for the more complex ones is that the number of possible logical moves becomes so large for harder problems that they could be solved equally well by random punching and that it seem to take a little longer time for the logical punching. Obviously, more investigation needs to be done with perhaps instructions being given on how many keys need to be punched for a solution and also how many areto be punched simultaneously. Some training in the appropriate specific logical procedures would undoubtedly be worthwhile. The interpretation of Table 4 is admittedly rather sketchy, but this discussion points out a possible conclusion and wherein the difficulties lie.

Some other suggestions regarding electromaze data and experiments might be made. It might be quite revealing to investigate the sequential order of punching that the different types of students apply to particular problems. This will reveal more than mere conjecture on the part of the observer or participants regarding the question of whether efficient systematic procedures are being applied. Secondly, now that the pilot studies have been initiated, we should take into consideration the correct sample sizes needed to protect against accepting false multivariate null hypotheses when certain alternative hypotheses are true. Two of the electromaze problems have proved able to contrast the two groups considered. The other problems may contrast groups of different academic or professional achievements or interest and this possibility might be investigated.

Summary

1. For logical and analytical reasons a certain multivariate analysis procedure was used to test the multivariate null hypothesis that two variates —trials and time— which relate to electromaze problem solution have equal population mean values for Physics and American Studies students at the University of Minnesota.

2. The test was derived from the observational model and the appropriate tests of significance were provided.

3. The complete detailed calculations were given for the first electromaze problem.

4. A table was provided which summarized the seven multivariate tests of significance and the separate tests of significance for each of the two variates considered independently.

5. Some comments were made which relate to interpreation of the data and for further experimentation.

References

1. Kruglak, Haym. "Electromaze Performance of Graduate Students in Physics and American Studies," Technical Report No. 3, ONR Project NR192-041. Physics Department, University of Minnesota, Minneapolis, 1952.
2. Moonan, William J. "The Generalization of the Principles of Modern Experimental Designs for Educational and Psychological Research," Thesis. Library of the University of Minnesota, Minneapolis, 216 pps., 1952.
3. Moonan, William J. "The Statistical Interpration of Degrees of Freedom," Journal of Experimental Education, March, 1953.